Metric and Topological spaces

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1 Introduction

This summary has been made by using "Introduction to Metric and Topological Spaces" second edition di Wilson a Sutherland.

2 Notation and Terminology

Theorem 1 (De Morgan Laws)

$$S \backslash \bigcup_I A_i = \bigcap (S \backslash A_i) \quad S \backslash \bigcap_I A_i = \bigcup_I (S \backslash A_i)$$

Definition 2 (Cartesian product) *The Cartesian product is the set of all ordered pairs* (a, b) *where* $a \in A$ *and* $b \in B$ *,*

$$A \times B = \{(a, b) \,|\, a \in A,\, b \in B\}$$

Definition 3 (injective function) *Let a map/function* $f : X \to Y$*, it said to be injective if* $\{f(x) = f(x') \Rightarrow x = x'\}$

Definition 4 (surjective or onto function) *Let a map/function* $f : X \to Y$*, it said to be onto if* $\{\forall y \in Y, \exists x \in X \,|\, y = f(x)\}$

Definition 5 (bijective function) *Let a map/function* $f : X \to Y$*, it said to be bijective if is both onto and injective.*

Definition 6 *Let* $f : X \to Y$*, where* $A \subseteq X$*, then* $f|_A : A \to Y$ *and is called restriction, i.e* $f|_A(a) = f(a)\ \forall a \in A$

Examples: Let C, D be subsets of a set X, then

$$(X \backslash C) \cap D = D \backslash C \quad C \backslash (D \cap C) = A \cap (X \backslash D)$$

3 More on sets and functions

Definition 7 (direct image) *Suppose* $f : X \to Y$ *be any map, and let* A *be subsets of* X*. The direct image* $f(A)$ *of* A *under* f *is the subset of* Y *given by*

$$\{y \in Y : y = f(a) \text{ for some } a \in A\}$$

Definition 8 (inverse image) *Suppose $f : X \to Y$ be any map, and let C be subsets of Y respectively. The inverse image $f^{-1}(C)$ of C under f is the subset of X given by $\{x \in X : f(x) \in C\}$*

Note that definition 8 does not require the existence of an inverse function.

Example: For any map $f : X \to Y$ we have $f(\emptyset) = \emptyset$ and $f^{-1}(\emptyset) = \emptyset$

Proposition 9 (3.6) *Suppose that $f : X \to Y$ is a map, that A, B are subsets of X and that C, D are subsets of Y. Then:*

$$f(A \cup B) = f(A) \cup f(B), \quad f(A \cap B) \subseteq f(A) \cap f(B)$$
$$f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D), \quad f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$$

Proposition 10 (3.7) *Suppose that $f : X \to Y$ is a map, and that for each i in some indexing set I we are given a subset A_i of X and a subset C_i of Y. Then:*

$$f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i), \quad f\left(\bigcap_{i\in I} A_i\right) \subseteq \bigcap_{i\in I} f(A_i)$$
$$f^{-1}\left(\bigcup_{i\in I} C_i\right) = \bigcup_{i\in I} f^{-1}(C_i), \quad f^{-1}\left(\bigcap_{i\in I} C_i\right) = \bigcap_{i\in I} f^{-1}(C_i)$$

Proposition 11 (3.8) *Suppose $f : X \to Y$ is a map and B, D are subsets of X, Y respectively. Then,*

$$f(X)\backslash f(B) \subseteq f(X\backslash B), \quad f^{-1}(Y\backslash D) = X\backslash f^{-1}(D)$$

Proposition 12 (3.9) *with the notation of Proposition 3.6,*

$$f(A)\backslash f(B) \subseteq f(A\backslash B), \quad f^{-1}(C\backslash D) = f^{-1}(C)\backslash f^{-1}(D)$$

Proposition 13 (3.13) *Suppose that $f : X \to Y$ is a map, $B \subseteq Y$ and for some indexing set I there is a family $\{A_i : i \in I\}$ of subsets of X with $X = \bigcup_I A_i$. Then*

$$f^{-1}(B) = \bigcup_I (f|A_i)^{-1}(B)$$

Proposition 14 (3.14) *Let X, Y be sets and $f : X \to Y$ a map. For any subsets $C \subseteq Y$ we have $f(f^{-1}(C)) = C \cap f(X)$. In particular, $f(f^{-1}(C)) = C$ if f is onto. For any subset $A \subseteq X$ we have $A \subseteq f^{-1}(f(A))$*

3.1 Inverse functions

Definition 15 (3.17) *A map $f : X \to Y$ is said to be invertible if there exists a map $g : Y \to X$ such that the composition $g \circ f$ is the identity map of X and the composition $f \circ g$ is the identity map of Y.*

Definition 16 (3.18) *A map $f : X \to Y$ is invertible if and only if it is bijective*

Proposition 17 (3.19) *When f is invertible, there is a uniqe g satisfing the definition.*

Proposition 18 (3.20) *Suppose that $f : X \to Y$ is one-one correspondence of sets X and Y and that $V \subseteq X$. Then the inverse image of V under the inverse map $f^{-1} : Y \to X$ equals the image set $f(V)$*

3.2 examples

Injectiviness: $f : X \to Y$ is a injective map if and only if $f(f^{-1}(C)) = C$ for all $C \subseteq Y$

Surjectiviness: $f : X \to Y$ is a surjective map if and only if $f^{-1}(f(A)) = A$ for all $A \subseteq X$

Ex. 3.8: Let $f : X \to Y$ be a map and let A, B be subsets of X, then $f(A \backslash B) = f(A) \backslash f(B)$ if and only if $f(A \backslash B) \cap f(B) = \emptyset$, i.e. f is injective.

4 Real Analysis

In here some brief review of real analysis. For more details see the summary of the real analysis course.

Definition 19 (4.2) *Given a non-empty subset S of $\mathbb{R}$ which is bounded above, we call u the least upper bound for S if*

a u is an upper bound for S

b $x \geq u$ for any upper bound x for S

Proposition 20 (4.4 Axiom of Completeness) *Any non-empty subset of $\mathbb{R}$ which is bounded above has a lest upper bound*

Remark: Although the completeness property was stated in terms of sets bounded above, it is equivalent to the corresponding property for sets bounded below

Proposition 21 (4.5) *If a non-empty subset S of $\mathbb{R}$ is bounded below then it has a greatest lower bound*

Proposition 22 (4.6) *The set $\mathbb{N}$ of positive integers is not bounded above*

Corollary 23 (4.7) *Between any two distinc real numbers x and y there is a rational number*

Remark: Between any two distinct real numbers there is also an irrational number

Proposition 24 (4.9 Triangle inequality)

$$|x + y| \leq |x| + |y| \quad \forall x, y \in \mathbb{R}$$

Corollary 25 (4.10 Inverse Triangle Inequality)

$$|x - y| \geq ||x| - |y|| \quad \forall x, y \in \mathbb{R}$$

4.1 Real sequences

Definition 26 (4.12) *The sequence (s_n) converges to (the real number) l if given (any real number) $\epsilon > 0$, there exists (an integer) N_ϵ such that $|s_n - l| < \epsilon$ for all $n \geq N$*

Remark: The smallest ϵ is, the larger N will need to be

Proposition 27 (4.13) *A convergent sequence has a unique limit*

Lemma 28 (4.14) *Suppose there is a positive real number K such that given $\epsilon > 0$ there exists N with $|s_n - l| < K\epsilon$ for all $n \geq N$. Then (s_n) convegres to l*

Definition 29 (4.15) *A sequence (s_n) is said to be monotonically increasing (decreasing) if $s_{n+1} \geq s_n$ $(s_{n+1} \leq s_n)$ for all $n \in \mathbb{N}$. It is is monotonic if it has either of these properties*

Theorem 30 (4.16) *Every bounded monotonic sequence of real numbers converges*

Definition 31 (4.17 Cauchy sequence) *A sequence (s_n) is a Cauchy sequence if given $\epsilon > 0$ there exists N such that if $m, n \geq N$ then $|s_m - s_n| < \epsilon$*

Definition 32 (4.18 Cauchy's convergence criterion) *A sequence (s_n) of real numbers converges if and only if it is a Cauchy sequence*

Theorem 33 (4.19 Bolzano-Weierstrass theorem) *Every bounded sequence of real numbers has at least one convegent subsequence*

Proposition 34 (4.20) *Suppose that $(s_n), (t_n)$ converge to s, t. Then*

- *a* $(s_n + t_n)$ *converges to* $s + t$
- *b* $(s_n t_n)$ *converges to* st
- *c* $1/t_n$ *converges to* $1/t$ *provided* $t \neq 0$

4.2 Limit functions

Definition 35 (4.21) *We say that $f(x)$ tends to the limit l as x tends to a if given (any real number) $\epsilon > 0$ there exists (a real number) $\delta > 0$ such that $|f(x) - l| < \epsilon$ for all real numbers x which satisfy $0 < |x - a| < \delta$*

Definition 36 (4.23) *The right hand limit $\lim_{x \to a^+} f(x)$ is equal to l if given $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all x in $(a, a + \delta)$*

Lemma 37 (4.25) *The following are equivalent:*

- *i* $\lim_{x \to a} f(x) = l$
- *ii* *if (x_n) is any sequence such that (x_n) converges to a but for all n we have $x_n \neq a$, then $f(x_n)$ converges to l*

4.3 Continuity

Definition 38 (4.26) *A function $f : \mathbb{R} \to \mathbb{R}$ has the intermidiate value property (IVP) if given any $a, b, d \in \mathbb{R}$ with $a < b$ and d between $f(a)$ and $f(b)$, there exists at least one c satisfying $a \leq c \leq b$ and $f(c) = d$*

Definition 39 (4.28) *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a if $\lim_{x \to a} f(x)$ exists and is $f(a)$*

Definition 40 (4.29) *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a if given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ for any x such that $|x - a| < \delta$*

Proposition 41 (4.30) *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ and that $f(a) \neq 0$. Then there exists $\delta > 0$ such that $f(x) \neq 0$ whenever $|x - a| < \delta$*

Proposition 42 (4.31) *Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are continuous at $a \in \mathbb{R}$. Then, so are: $|f|$, $f + g$, fg and if $g(a) \neq 0$ then $1/g$ is continuous at a*

Proposition 43 (4.32) *Let $p : \mathbb{R} \to \mathbb{R}$ be the polynomial function, then p is continuous.
Let $r : \mathbb{R} \backslash Z \to \mathbb{R}$ be the rational function $x \mapsto p(x)/q(x)$ where p, q are polynomial functions and Z is the zero set of q. Then, r is continuous on $\mathbb{R} \backslash Z$*

Proposition 44 (4.33) *Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are such that f is continuous at $a \in \mathbb{R}$, and g is continuous at $f(a)$. Then, $g \circ f$ is continuous at a*

Definition 45 (4.34) *Let $f : X \to \mathbb{R}$ be a function defined on a subset $X \subseteq \mathbb{R}$ and let $a \in X$. We say f is continous at a if given $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$ and $x \in X$*

Theorem 46 (4.35) *The intermideo value property holds also for continuous functions $f : I \to \mathbb{R}$ for any interval $I \in \mathbb{R}$*

5 Metric Spaces

The motivation of metric spaces comes from studying continuity.

Definition 47 (5.1) *A function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous at a point $a \in \mathbb{R}^n$, say $a = (a_1, a_2, ..., a_n)$, if given $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ for every $x = (x_1, x_2, ..., x_n)$ satisfying*

$$\sqrt{\left(\sum_{i=1}^{n} (x_i - a_i)^2\right)} < \delta$$

Definition 48 (5.2) *A metric space consists of a non-empty set X together with a function $d : X \times X \to \mathbb{R}$ such that the following holds:*

M1 For all $x, y \in X$, $d(x, y) \geq 0$; and $d(x, y) = 0$ iff $x = y$
M2 (Symmetry) for all $x, y \in X$, $d(y, x) = d(x, y)$
M3 For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$

The elements of X are called points of the space, and d is called the metric or the distance function

Definition 49 (5.3) *Suppose that (X, d_X) and (Y, d_Y) are metric spaces and let $f : X \to Y$ be a map*

- *We say f is continuous at $x_0 \in X$ if given $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \epsilon$ whenever $d_X(x, x_0) < \delta$*
- *We say f is continuous if f is continuous at every $x_0 \in X$*

Definition 50 (Metric subspaces) *Suppose that (X, d) is a metric space and that A is a non-empty subset of X. Let $d_A : A \times A \to \mathbb{R}$ be the restriction of d to $A \times A$, then the metric axiom hold for d_A since they hold for d and the metric space (A, d_A) is called a metric subspace of (X, d)*

Definition 51 (Discrete metric) *Let X be any non-empty set and define d_0 by*

$$d_0(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

Definition 52 (Manhattan/city block metric) *Let $X = \mathbb{R}^2$ and for $x = (x_1, x_2)$, $y = (y_1, y_2)$ and define d_1 by*

$$d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$$

Definition 53 (Euclidean n-space) *The Euclidean n-space $(\mathbb{R}^n, d_2)$ where for $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ then*

$$d_2(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Definition 54 (penalty) *Let $X = \mathbb{R}^n$ and for $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ and define d_∞ by*

$$d_\infty(x, y) = \max_{i=1}^{n} \{|x_i - y_i|\}$$

Definition 55 (Supremum metric) *Let X be the set of all bounded functions $f : [a, b] \to \mathbb{R}$. Given two points f and g in X, let*

$$d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$$

5.1 Continuous functions on metric spaces

Proposition 56 (5.17) *Suppose $f, g : X \to \mathbb{R}$ are continous real-valued functions on a metric space (X, d). Then so are: $|f|$, $f + g$, fg and also if g is never zero on X, then $1/g$ is continuous on X*

Proposition 57 (5.18) *Suppose that $f : X \to Y$ and $g : Y \to Z$ are maps of metric spaces with metrics d_X, d_Y, d_Z that f is continuous at $a \in X$ and g is continuous at $f(a)$. Then $g \circ f$ is continuous at a*

Proposition 58 (5.19) *Suppose that $f : X \to X'$, $g : Y \to Y'$ are maps of metric spaces which are continuous at $a \in X$, $b \in Y$ respectively. Then the map $f \times g : X \times Y \to X' \times Y'$ given by $(f \times g)(x, y) = (f(x), g(y))$ for all $(x, y) \in X \times Y$, is continuous at (a, b)*

Proposition 59 (5.20) *The projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ of a metric product onto its factors, defiend by $p_X(x, y) = x$ and $p_Y(x, y) = y$ are continuous*

Definition 60 (5.21) *The diagonal map $\Delta : X \to X \times X$ of any set X is the map defined by $\Delta(x) = (x, x)$*

Proposition 61 (5.22) *The diagonal map of any metric space is continuous*

5.2 Bounded sets in metric spaces

Definition 62 (5.23) *A subset S of a metric space (X, d) is bounded if there exist $x_0 \in X$ and $K \in \mathbb{R}$ such that $d(x, x_0) \leq K$ for all $x \in S$*

Note that x_0 can not belong to the set S

Definition 63 (5.24) *If S is a non-empty bounded subset of a metric space with metric d, then the diameter of S is $\sup\{d(x, y) : x, y \in S\}$. The diameter of the empty set is 0*

Definition 64 (5.25) *If $f : S \to X$ is a map from a set S to a metric space X, then we say f is bounded if the subset $f(S)$ of X is bounded*

Proposition 65 (5.26) *The union of any finite number of bounded subsets of a metric space is bounded*

$$S = \bigcup_{i=1}^{N<\infty} S_i \subseteq X; \quad S_i \text{ bounded } \Rightarrow S \text{ bounded}$$

5.3 Open balls in metric spaces

Definition 66 (5.27) *Let (X, d) be a metric space, $x_0 \in X$, and $r > 0$ a real number. The open ball X of radius r centred on x_0 is the set*

$$B_r(x_0) = \{x \in X : d(x, x_0) < r\}$$

if we are considering more than one metric on X then we write $B_r^d(x_0)$

Definition 67 (Reformulation of bounded) *Let the metric space (X, d_X), where $S \subseteq X$ and $r \in \mathbb{R} > 0.$, then the subset S of the metric space X is bounded if and only if $S \subseteq B_r^{d_X}(x_0)$ for some $x_0 \in X$.*

$$S \text{ bounded } \Leftrightarrow \exists r > 0, \exists x_0 \in X \mid S \subseteq B_r^{d_X}(x_0)$$

Proposition 68 (5.30) *With notation as in definition 5.3, f is continuous at x_0 iff given $\epsilon > 0$ there exists $\delta > 0$ such that $f(B_\delta^{d_X}(x_0)) \subseteq B_\epsilon^{d_Y}(f(x_0))$*

Proposition 69 (5.31) *Given an open ball $B_r(x)$ in a metric space (X, d) and a point $y \in B_r(x)$, there exits $\epsilon > 0$ such that $B_\epsilon(y) \subseteq B_r(x)$*

5.4 Open sets in metric space

Definition 70 (5.32) *Let (X, d) be a metric space and $U \subseteq X$. We say that U is open in X if for every $x \in U$ there exists $\epsilon_x > 0$ such that $B_{\epsilon_x}(x) \subseteq U$*

Proposition 71 (5.37) *Suppose that $f : X \to Y$ is a map of metric spaces. Then f is continuous iff $f^{-1}(U)$ is open in X whenever U is open in Y*

Proposition 72 (5.39) *If $U_1, U_2, ..., U_m$ are open in a metric space X then is $\bigcap_{i=1}^{m} U_i$*

Proposition 73 (5.41) *The union of any collection of sets open in a metric space X is open in X*

5.5 Examples

Bound subset of a metric space: Suppose that (X, d) is a metric space, $A \subset X$, then A is bounded if and only if there is some constant Δ such that $d(a, a') \leq \Delta$ for all $a, a' \in A$.

Diameters: Suppose that $A \subseteq B$ where B is bounded subset of a metric space. Then A is bounded and $diam\, A \leq diam B$

Union of open balls: A subset of a metric space is open if and only if is a union of open balls.

6 More concepts in metric spaces

Definition 74 (6.1) *A subset V of a metric space X is closed in X if $X \backslash V$ is open in X*

Proposition 75 (6.3) *If $V_1, ..., V_m$ are closed subsets of a metric space X, then so is $\bigcup_{i=1}^{m} V_i$*

Proposition 76 (6.4) *The intersection of any family of sets each of which is closed in a metric space X is also closed in X*

Proposition 77 (6.5) *For any metric space X, the empty set $\emptyset$ and the whole set X are closed in X*

Proposition 78 (6.6) *Let X, Y be metric spaces and let $f : X \to Y$ be a map. Then f is continuous iff $f^{-1}(V)$ is closed in X whenever V is closed in Y*

6.1 Closure

Definition 79 (6.7) *Suppose that A is a subset of a metric space X, and $x \in X$. we say that x is a point of closure of A in X if given $\epsilon > 0$ we have $B_\epsilon(x) \cap A \neq \emptyset$. The closure of A in X is the set of all points of closure of A in X*

When it is agreed which metric space X we are taking closure in, we denote the closure of A in X by $\overline{A}$

Definition 80 (6.9) *A subset A of a metric space X is said to be dense in X if $\overline{A} = X$*

Proposition 81 (6.11) *Let A, B be subsets of a metric space X. Then*

1. $A \subseteq \overline{A}$
2. $A \subseteq B$ *implies that* $\overline{A} \subseteq \overline{B}$
3. A *is closed in* X *if and only if* $\overline{A} = A$
4. $\overline{\overline{A}} = \overline{A}$
5. $\overline{A}$ *is closed in* X
6. $\overline{A}$ *is the smallest closed subset of* X *containing* A

Proposition 82 (6.12) *A map $f : X \to Y$ of a metric spaces is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for every $A \subseteq X$*

Proposition 83 (6.13) *Let $A_1, ..., A_m$ be subsets of a metric space X. Then*

$$\overline{\bigcup_{i=1}^{m} A_i} = \bigcup_{i=1}^{m} \overline{A}_i$$

Proposition 84 (6.14) *For each i in some indexing set I, let A_i be a subset of the metric space X. Then*

$$\overline{\bigcap_{i\in I} A_i} \subseteq \bigcap_{i\in I} \overline{A}_i$$

6.2 Limit points

Definition 85 (6.15) *A point x in a metric space X is said to be a limit point of a subset A in X if given $\epsilon > 0$ there is a point in $B_\epsilon(x) \cap A$ other than x itself, i.e. $(B_\epsilon(x) \backslash \{x\}) \cap A \neq \emptyset$*

Proposition 86 (6.17) *A subset A of a metric space X is closed in X iff it contains all its limit points in X*

Proposition 87 (6.18) *Let A be any subset of a metric space X. Then, $\overline{A}$ is the union of A with all its limits points in X*

6.3 Interior

Definition 88 (6.19) *The interior* $\mathrm{Int}(A)$ *of a subset A in a metric space X is the set of points $a \in A$ such that $B_\epsilon(a) \subseteq A$ for some $\epsilon > 0$*

Proposition 89 (6.21) *Let A, B be subsets of a metric space X. Then*

1. $\mathrm{Int}(A) \subseteq A$
2. $A \subseteq B$ *implies that* $\mathrm{Int}(A) \subseteq \mathrm{Int}(B)$
3. A *is open in* X *iff* $\mathrm{Int}(A) = A$
4. $\mathrm{Int}(\mathrm{Int}(A)) = \mathrm{Int}(A)$
5. $\mathrm{Int}(A)$ *is open in* X
6. $\mathrm{Int}(A)$ *is the largest open subset of* X *contained in* A

6.4 Boundary

Definition 90 (6.22) *The boundary ∂A of a subset A in a metric space X is the set $\overline{A} \setminus \operatorname{Int}(A)$*

Proposition 91 (6.24) *Given a subset A of a metric space X, a point $x \in X$ is in ∂A iff for every $\epsilon > 0$ both $A \cap B_\epsilon(x)$ and $(X \setminus A) \cap B_\epsilon(x)$ are non-empty*

6.5 Convergence in metric space

Definition 92 (6.25) *A sequence (x_n) in a metric space X converges to a point $x \in X$ if given (any real number) ($\epsilon > 0$, there exists (an integer) N such that $x_n \in B_\epsilon(x)$ whenever $n \geq N$*

Proposition 93 (6.26) *Suppose that a sequence (x_n) in a metric space (X, d) converges to x and also to y in X. Then $x = y$*

Definition 94 (6.27) *A sequence (x_n) in a metric space (X, d) is called a Cauchy sequence if for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ whenever $m, x \geq N$*

Proposition 95 (6.28) *Ay convergent sequence in a metric space is a Cauchy sequence*

Proposition 96 (6.29) *Suppose Y is a subset of a metric space X and that (y_n) is a sequence in Y which converges to a point $a \in X$. Then $a \in \overline{Y}$*

Corollary 97 (6.30) *If Y is a closed subset of a metric space X and (y_n) is a sequence of points in Y which converges in X to a point a then $a \in Y$*

6.6 Examples

Ex. 6.9: If A is a non-empty bounded subset of $\mathbb{R}$ then $\sup A$ and $inf\ A$ are in $\overline{A}$

Ex. 6.10: If A is a bounded subset of a metric space then $\overline{A}$ is bounded and $diam\,\overline{A} = diam\,A$
Ex. 6.23 For a subset of a metric space X, the following holds

- $\operatorname{Int}(A) = A \setminus \partial A = \overline{A} \setminus \partial A$
- $\overline{X \setminus A} = X \setminus \operatorname{Int}(A)$
- $\partial A = \overline{A} \cap \overline{X \setminus A} = \partial(X \setminus A)$
- ∂A is closed in X

7 Topological spaces

Definition 98 *A topological space $T = (X, \mathcal{T})$ consists of a non-empty set X together with a fix family $\mathcal{T}$ of subsets of X satisfying*

T1 $X, \emptyset \in \mathcal{T}$
T2 the intersection of any two sets in $\mathcal{T}$ is in $\mathcal{T}$
T3 the union of any collection of sets in $\mathcal{T}$ is in $\mathcal{T}$

Remark: it is important to remember that $\mathcal{T}$ is in general only a subfamily of the family of all subsets of X

Proposition 99 (7.2) *For a subset U of a topological space X to be open in X it is necessary and sufficient that for every $x \in U$ there is an open subset U_x of X such that $x \in U_x \subseteq U$*

Definition 100 (7.6) *Given two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on the same set, we say that $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$.*
The opposite of coarser is finer, we say $\mathcal{T}_2$ is finer than $\mathcal{T}_1$ iff $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$

7.1 Examples

Remark: Given a metric space (X, d) we may construct a topological space $(X, \mathcal{T}_d)$ by defining $\mathcal{T}_d$ to consist of precisely those subsets of X which are $d-$open. This topological space that arises from a metric space is called metrizable.

Remark: The metrics d_1, d_2, d_∞ on $\mathbb{R}^n$ all give rise to the same open sets and hence to the same topology (The Euclidean topology)

Indiscrete topology: Let X be any non-empty set. The indiscrete topology on X is the family $\{\emptyset, X\}$.

Co-finite topology: Let X be any non-empty set. The co-finite topology on X consists of the empty set together with every subset U of X such that $X \backslash U$ is finite.

Intersection and Union of topologies: The intersection of topologies on the same set is also a topology on the set. The union of topologies, instead, may or may not be a topology.

8 Continuity in topological spaces; bases

Definition 101 (8.1) *We say that a map $f : X \to Y$ of topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is continuous if $U \in \mathcal{T}_Y \Rightarrow f^{-1}(U) \in \mathcal{T}_X$ if necessary clarity we say that f is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous*

Definition 102 (8.2) *With the notation of definition 8.1, we say that f is continuous at a point $x \in X$ if, given any $U' \in \mathcal{T}_Y$ such that $f(x) \in U'$, there is some $U \in \mathcal{T}_X$ such that $x \in U$ and $f(U) \subseteq U'$*

Proposition 103 (8.3) *If (X, d_X), (Y, d_Y) are metric spaces whose underlying topological spaces are $(X, \mathcal{T}_X)$, $(Y,$*
$T_Y)$, then a map $f : X \to Y$ is (d_X, d_Y)-continuous iff is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous

Definition 104 (8.4) *Given spaces $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, $(Z, \mathcal{T}_Z)$ and continuous maps $f : X \to Y$, $g : Y \to Z$, the composition of $g \circ f : X \to Z$ is continuous (more precisely, if f is $(\mathcal{T}_X, \mathcal{T}_Y)$ continuous and g is $(\mathcal{T}_Y, \mathcal{T}_Z)$ continuous then $g \circ f$ is $(\mathcal{T}_X, \mathcal{T}_Z)$ continuous)*

Proposition 105 (8.6) *1. The identity map of any topological space is continuous;*

2. *If $\mathcal{T}_X$ is the discrete topology than any map $f : X \to Y$ to another topological space $(Y, \mathcal{T}_Y)$ is continuous;*

3. *If $\mathcal{T}_Y$ is the indiscrete topology than ay map $f : X \to Y$ from another topological space $(X, \mathcal{T}_X)$ is continuous*

8.1 Homeomorphisms

Definition 106 (8.7) *A homeomorphism between topological spaces X and Y is a bijective map $f : X \to Y$ such that f and its inverse function f^{-1} are both continuous*

Remark: If a homeomorphism exists between spaces we say that they are homeomorphic or just equivalent

8.2 Bases

Definition 107 (8.9) *Given a topological space $(X, \mathcal{T})$, a basis for $\mathcal{T}$ is a subfamily $\mathcal{B} \subseteq \mathcal{T}$ such that every set in $\mathcal{T}$ is a union of sets from $\mathcal{B}$*

Proposition 108 (8.12) *To check that a map $f : X \to Y$ of topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is continuous, it is enough to check that for each open set B in some basis for $\mathcal{T}_Y$, the inverse image $f^{-1}(B)$ is open in X*

Remark: A topological space which admits a countable basis for open sets is called second countable.

9 Some concepts in topological spaces

Definition 109 (9.1) *Let $(X, \mathcal{T})$ be a topological space. A subset V of X is said to be closed in X if $(X \backslash V$ is open in X*

Proposition 110 (9.4) *Let X be a topological space. Then*

1. *$\emptyset, X$ are closed in X*

2. *if V_1, V_2 are closed in X then $V_1 \cup V_2$ is closed in X*

3. *if V_i is closed in X for all $i \in I$ then $\bigcap_{i \in I} V_i$ is closed in X*

Proposition 111 (9.5) *A map $f : X \to Y$ of topological spaces is continuous iff $f^{-1}(V)$ is closed in X whenever V is closed in Y*

Definition 112 (9.6) *A point a is a point of closure of a subset A in a topological space X if $U \cap A \neq \emptyset$ for any open U of X with $a \in U$. The closure of $\overline{A}$ of A in X is the set of points of closure of A in X*

Definition 113 (9.9) *A subset A of a topological space X is said to be dense in X if $\overline{A} = X$*

Proposition 114 (9.10) *Let A, B be subsets of a topological space X. Then*

1. $A \subseteq \overline{A}$
2. $A \subseteq B$ *implies that* $\overline{A} \subseteq \overline{B}$
3. A *is closed in* X *if and only if* $\overline{A} = A$
4. $\overline{\overline{A}} = \overline{A}$
5. $\overline{A}$ *is closed in* X
6. $\overline{A}$ *is the smallest closed subset of* X *containing* A

Proposition 115 (9.11) *A map* $f : X \to Y$ *of a topological spaces is continuous if and only if* $f(\overline{A}) \subseteq \overline{f(A)}$ *for every* $A \subseteq X$

Proposition 116 (9.12) *Let* $A_1, ..., A_m$ *be subsets of a topological space* X. *Then*

$$\overline{\bigcup_{i=1}^{m} A_i} = \bigcup_{i=1}^{m} \overline{A}_i$$

Proposition 117 (9.13) *For each* i *in some indexing set* I, *let* A_i *be a subset of the topological space* X. *Then*

$$\overline{\bigcap_{i\in I} A_i} \subseteq \bigcap_{i\in I} \overline{A}_i$$

Definition 118 (9.14) *A point* a *is an interior point of a subset* A *of a topological space* X *if there exists some set* U *which is open in* X *and with* $a \in U \subseteq A$. *The set of all interior points of* A *is called interior of* A.

Proposition 119 (9.16) *We have* $\overline{X \backslash A} = X \backslash \operatorname{Int}(A)$ *for any subset* A *of a space* X

Proposition 120 (9.17) *Let* A, B *be subsets of a topological space* X. *Then*

1. $\operatorname{Int}(A) \subseteq A$
2. $A \subseteq B$ *implies that* $\operatorname{Int}(A) \subseteq \operatorname{Int}(B)$
3. A *is open in* X *iff* $\operatorname{Int}(A) = A$
4. $\operatorname{Int}(\operatorname{Int}(A)) = \operatorname{Int}(A)$
5. $\operatorname{Int}(A)$ *is open in* X
6. $\operatorname{Int}(A)$ *is the largest open subset of* X *contained in* A

Definition 121 (9.18) *The boundary* ∂A *of a subset* A *of a space* X *is the set* $\overline{A} \backslash \operatorname{Int}(A)$

Proposition 122 (9.20) *The boundary of a subset* A *in a space* X *is* $\overline{A} \cap \overline{X \backslash A}$

Corollary 123 (9.21) *We have* $\partial A = \partial(X \backslash A)$ *for any subset* A *of a space* X

Definition 124 (9.22) *A neighbourgh of a point* x *in a space* X *is a subset* N *of* X *which contains an open subset of* X *containing* x

Remarks: Suppose A is a subset of a space X, then

- A is closed in X if and only if $\partial A \subseteq A$
- $\partial A = \emptyset$ if and only if A is open and closed in X

10 Subspaces and product spaces

10.1 Subspaces

Definition 125 (10.3) *Let $(X, \mathcal{T})$ be a topological space and let A be a non-empty subset of X. The subspace topology on A is $\mathcal{T}_A = \{A \cap U : U \in \mathcal{T}\}$*

Proposition 126 (10.4) *Let $(X, \mathcal{T})$ be a topological space and let A be a non-empty subset of X with the subspace topology $\mathcal{T}_A$. Then the inclusion map $i : A \to X$ defined by $i(a) = a$ for all $a \in A$, is $(\mathcal{T}_A, \mathcal{T})$ continuous*

Corollary 127 (10.5) *Let $f : X \to Y$ be a continuous map of topological spaces $(X, \mathcal{T})$, $(Y, \mathcal{T}')$ and let A be a non-empty subset of X with the subspace topology $\mathcal{T}_A$. Then the restriction $f|_A : A \to Y$ is $(\mathcal{T}_A, \mathcal{T}')$ continuous*

Proposition 128 (10.6) *Let X be a topological space, let A be a subspace of X and let $i : A \to X$ be the inclusion map. Suppose that Z is a topological space and that $g : Z \to A$ is a map. Then g is continuous iff $i \circ g : Z \to X$ is continuous*

Proposition 129 (10.8) *With notation as in proposition 10.6, the subspace topology $\mathcal{T}_A$ on A is the only topology satisfying proposition 10.6 for all possible maps g*

10.2 Products

Proposition 130 (10.9) *Suppose $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ are topological spaces, and let $\mathcal{T}_{X\times Y}$ be the family of all unions of sets of the form $U \times V$ where $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$. Then $\mathcal{T}_{X\times Y}$ is a topology for $X \times Y$, called product topology*

Proposition 131 (10.10) *The two projection maps $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are continuous, where $p_X(x, y) = x$ and $p_Y(x, y) = y$ for all $(x, y) \in X \times Y$*

Proposition 132 (10.11) *Any map $f : Z \to X \times Y$ from a topological space Z into the topological product $X \times Y$ is continuous if and only if both $p_X \circ f : Z \to X$ and $p_Y \circ f : Z \to Y$ are continuous*

Proposition 133 (10.12) *If $f : X \to X'$ and $g : Y \to Y'$ are continuous, then so if $f \times g : X \times Y \to X' \times Y'$ defined by $(f \times g)(x, y) = (f(x), g(x))$*

Proposition 134 (10.13) *For any topological space X let $\Delta : X \to X \times X$ be the diagonal map defined by $\Delta(x) = (x, x)$. Then Δ is continuous*

Proposition 135 (10.14) *Let X and Y be topological spaces, and let $y_0 \in Y$. Define $i_{y_0} : X \to X \times Y$ by $i_{y_0}(x) = (x, y_0)$. Then i_{y_0} is continuous*

Proposition 136 (10.15) *If $f, g : X \to \mathbb{R}$ are continuous real-valued functions on a topological space X then so are: $|f|$, $f + g$, fg and also if g is never zero on X then $1/g$ is also continuous*

Proposition 137 (10.20) *$W \subseteq X \times Y$ is open in $X \times Y$ if and only if for any $(x, y) \in W$ there exists subsets U, V of X, Y respectively which are open in X, Y and with $(x, y) \in U \times V \subseteq W$*

11 The Hausdorff condition

Definition 138 (11.1) *A sequence of points (x_n) in a topological space X converges to a point $x \in X$ if given any open set $U \ni X$ there exists (an integer) N such that $x_n \in U$ whenever $n \geq N$*

Definition 139 (11.3) *A topological space X satisfies the Hausdorff condition if for any two distinct points $x, y \in X$ there exist disjoint open set U, V of X such that $x \in U$, $y \in V$*

Remark: We refer to a topological space which satisfies the Hausdorff condition as a Hausdorff space

Remark: Let $x_1, ..., x_n$ be distinct points in a Hausdorff space X. Then there exists pairwise disjoint open subsets $U_1, ..., U_n$ of X such that $x_i \in U_i$ for every $i \in \{1, ..., n\}$

Proposition 140 (11.4) *In a Hausdorff space, any given convergent sequence has a unique limit*

Proposition 141 (11.5) *Any metrizable space $(X, \mathcal{T})$ is Hausdorff*

Proposition 142 (11.7) *a Any subspace of a Hausdorff space is Hausdorff.*

b The topological product $X \times Y$ of spaces X and Y is Hausdorff if and only if both X and Y are Hausdorff.

- *if $f : X \to Y$ is an injective continuous map of topological spaces and Y is Hausdorff then so is X*
- *If spaces X and Y are homeomorphic then X is Hausdorff if and only if Y is Hausdorff.*

Definition 143 (11.8) *A topological space is regular (normal) if given any closed subset $V \subset X$ and point $x \in X \backslash V$ there exist disjoint open subsets U, U' of X such that $V \subset U$ and $x \in U'$*

12 Connected spaces

Definition 144 (12.1) *A topological space X is connected if there does not exist a continuous map from X onto a two-point discrete space.*

Remark: X is connected if any continuous map from X to a two-point discrete space is constant

Definition 145 (12.2) *A partition $\{A, B\}$ of a topological space X is a pair of non-empty subsets A, B of X such that $X = A \cup B$, $A \cap B = \emptyset$, and both A, B are open in X*

Remark: A and B are also closed in X, and $(A \cap \overline{B}) \cup (\overline{A} \cap B) = \emptyset$

Proposition 146 (12.3) *A topological space is connected if and only if admits no partitions*

Corollary 147 (12.4) *A topological space X is connected if and only if the only subsets of X which are both open and closed in X are $X, \emptyset$*

Remark: Any discrete space with at least two points is disconnected.

Remark: Any indiscrete space is connected

Definition 148 (12.6) *A non-empty subset A of a topological space X is connected if A with the subspace topology is connected. Conventionally we regard the empty set to be connected.*

Proposition 149 (12.7) *A non-empty subset $S \subseteq \mathbb{R}$ is an interval if and only if it satisfies the following property: if $x, y \in S$ and $z \in \mathbb{R}$ are such that $x < z < y$ then $z \in S$*

Theorem 150 (12.8) *Any connected subspace S of $\mathbb{R}$ is an interval*

Theorem 151 (12.10) *Any interval I in $\mathbb{R}$ is connected*

Proposition 152 (12.11) *Suppose that $f : X \to Y$ is a continuous map of topological spaces and that X is connected. Then $f(X)$ is connected*

Corollary 153 (12.12) *Connectedness is a topological property*

Corollary 154 (12.14) *Suppose $f : X \to \mathbb{R}$ is continuous and X is connected. Then $f(X)$ is an interval.*

Corollary 155 (12.15 Intermediate value theorem) *If $f : [a, b] \to \mathbb{R}$ is continuous then it has the intermediate value property*

Proposition 156 (12.16) *Suppose $\{A_i : i \in I\}$ is an indexed family of connected subsets of a topological space X with $A_i \cap A_j \neq \emptyset$ for each pair $i, j \in I$. Then $\bigcup_{i \in I} A_i$ is connected*

Corollary 157 (12.17) *Suppose that $\{C_i : i \in I\}$ and B are connected subsets of a space X such that for every $i \in I$ we have $C_i \cap B \neq \emptyset$. Then $B \cup \bigcup_{i \in I} C_i$ is connected*

Theorem 158 (12.18) *The topological product $X \times Y$ of spaces X, Y is connected if and only if X, Y are connected.*

Proposition 159 (12.19) *Suppose that A is a connected subset of a space X and that $A \subseteq B \subseteq \overline{A}$. Then B is connected*

12.1 Path-connectedness

Definition 160 (12.20) *For points x, y in a topological space X, a path in X from x to y is a continuous map $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. We say that such a path joins x and y*

Definition 161 (12.21) *A topological space X is path-connected if any two points of X can be joined by a path in X*

Proposition 162 (12.23) *Any path-connected space X is connected*

Proposition 163 (12.25) *A connected open subset U of $\mathbb{R}^n$ is path-connected*

13 Compact spaces

Proposition 164 (13.1) *A continuous function* $f : [a, b] \to \mathbb{R}$ *is bounded on* $[a, b]$

Definition 165 (13.3) *Suppose* X *is a set and* $A \subseteq X$*. A family* $\{U_i : i \in I\}$ *of subsets of* X *is called a cover for* A *if* $A \subseteq \bigcup_{i \in I} U_i$

Definition 166 (13.4) *A subcover of a cover* $\{U_i : i \in I\}$ *for* A *is a subfamily* $\{U_j : j \in J\}$ *for some subset* $J \subseteq I$ *such that* $\{U_j : j \in J\}$ *is still a cover for* A*. We call it a finite subcover if* J *is finite*

Definition 167 (13.5) *If* $\mathcal{U} = \{U_i : i \in I\}$ *is a cover for a subset* A *of a topological space* X *and if each* U_i *is open in* X *then* $\mathcal{U}$ *is called an open cover for* A

Definition 168 (13.6) *A subset* A *of a topological space* X *is compact if every open cover for* A *has a finite subcover*

Remark: every open interval in $\mathbb{R}$ with the usual topology is not compact.

Remark: Any finite subset of a space X is compact

Remark: Any space with the co-finite topology is compact

13.1 Compactness of closed bounded intervals

Theorem 169 (13.9) *Any closed bounded interval* $[a, b]$ *in* $\mathbb{R}$ *is compact*

13.2 Properties of compact spaces

Proposition 170 (13.10) *Any compact subset* C *of a metric space* (X, d) *is bounded*

Corollary 171 (13.11) *Any compact subset of* $\mathbb{R}^n$ *is bounded*

Proposition 172 (13.12) *Let* C *be a compact subset of a Hausdorff space* X*. Then* C *is closed in* X

Remark: If C, C' are compact subsets of a Hausdorff space X then $C \cap C'$ is compact

Corollary 173 (13.13) *Any compact subset of* $\mathbb{R}^n$ *is closed in* $\mathbb{R}^n$

Definition 174 (13.14) *A subset* A *of a topological space* X *is said to be relatively compact in* X *if* $\overline{A}$ *is compact, where the closure is taken in* X

13.3 Continuous maps on compact spaces

Proposition 175 (13.15) *If* $f : X \to Y$ *is a continuous map of topological spaces and* X *is compact then* $f(X)$ *is compact*

Corollary 176 (13.16) *Compactness is a topological property*

Corollary 177 (13.17) *Any continuous map from a compact space to a metric space is bounded*

Corollary 178 (13.18) *If* $f : C \to \mathbb{R}$ *is continuous and* C *is compact then* f *attains its bound on* C*. This means there is at least one* $c_0 \in C$ *such that* $f(c_0) = inf\, f(C)$ *and at least one* $c_1 \in C$ *such that* $f(c_1) = \sup f(C)$

Corollary 179 (13.19) *A continuous real-valued function on* $[a, b]$ *attains its bounds*

13.4 Compactness of subspaces and products

Proposition 180 (13.20) *Any closed subset* C *of a compact space* X *is compact*

Theorem 181 (13.21) *A topological product* $X \times Y$ *of spaces* X, Y *is compact if and only if both* X *and* Y *are compact.*

Theorem 182 (13.22 Heine-Borel theorem) *Any closed bounded subset* C *of* $\mathbb{R}^n$ *is compact*

Remark: A subset of a metric space may be bounded and closed without being compact

13.5 Compactness and uniform continuity

Definition 183 (13.23) *A map* $f : X \to Y$ *of a metric spaces* X, Y *with metrics* d_X, d_Y *is said to be uniformly continuous on* X *if given* $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $d_Y(f(x), f(a)) < \epsilon$ *for any* $x, a \in X$ *satisfying* $d_X(x, a) < \delta$

Remark: Notice that this is a stronger than ordinary continuity in that δ can depend on ϵ but not on a (i.e uniformly).
Ordinary continuity of f is a local property in that it says something about the behaviour of f in some neighbourhood of each point in X.
Uniform continuity is a global property since it says something about the behaviour of f over the whole space X.

Proposition 184 (13.24) *If* $f : X \to Y$ *is a continuous map of metric spaces and* X *is compact then* f *is uniformly continuous on* X

13.6 An inverse function theorem

Proposition 185 (13.26) *Suppose that* $f : X \to Y$ *is a continuous one-one correspondence, where* X *is a compact space and* Y *is a Hausdorff space. Then* f *is a homeomorphism.*

Corollary 186 (13.27) *if* $f : X \to Y$ *is a continuous injective map from a comapct space* X *into a Hausdorff space* Y*, then* f *determines a homeomorphism of* X *onto* $f(X)$

14 Sequential compactness

14.1 Sequential compactness for real numbers

Definition 187 (14.2) *A subset $S \subseteq \mathbb{R}$ is called sequentially compact if every sequence in S has at least one subsequence converging to a point in S*

Proposition 188 (14.3) *Any closed bounded subset $S \subseteq \mathbb{R}$ is sequentially compact*

Theorem 189 (14.5) *A subset $S \subseteq \mathbb{R}$ is sequentially compact if and only if it is bounded and closed in $\mathbb{R}$*

Theorem 190 (14.6) *A subset of $\mathbb{R}$ is compact if and only if it is sequentially compact*

14.2 Sequential compactness for metric spaces

Definition 191 (14.7) *A metric space X is sequentially compact if every sequence in X has at least one subsequence converging to a point of X*

Definition 192 (14.8) *A non-empty subset A of a metric space (X, d) is sequentially compact if, with the subspace metric d_A, it satisfies the definition. Conventionally the empty set is considered to be sequentially compact*

Remark: Any finite metric space is sequentially compact

Theorem 193 (14.10) *A metric space is compact if and only if it is sequentially compact*

Proposition 194 (14.11) *Let (x_n) be a sequence in a metric space X and let $x \in X$. Suppose that for each $\epsilon > 0$ the neighbourhood $B_\epsilon(x)$ contains x_n for infinitely many values of n. Then (x_n) has a subsequence converging to x*

Remark: Notice that the condition says that $B_\epsilon(x)$ contains x_n for infinitely many values of n, not that it contains ifinitely many different points in the set $\{x_n : n \in \mathbb{N}\}$.

Corollary 195 (14.13) *Suppose that a sequence (x_n) in a metric space X has no convergent subsequences. Then for each $x \in X$ there exists $\epsilon_x > 0$ such that $B_{\epsilon_x}(x)$ contains x_n for only finitely many values of n*

Theorem 196 (14.15) *Any compact subset X of a metric space Y is sequentially compact*

Definition 197 (14.16) *Let $\mathcal{U}$ be any family of subsets of a metric space X covering a subset $A \subseteq X$. A Lebesgue number for $\mathcal{U}$ is a real number $\epsilon > 0$ such that for any $a \in A$ the ball $B_\epsilon(a)$ is contained in some single set from $\mathcal{U}$.*

Proposition 198 (14.18) *Any open cover $\mathcal{U}$ of a sequentially compact metric space X has a Lebesgue number*

Definition 199 (14.19) *Given a real number $\epsilon > 0$ and a metric space X, a subset $N \subseteq X$ is called an $\epsilon-$net for X if the family $\{B_\epsilon(x) : x \in \mathbb{N}\}$ covers X*

Proposition 200 (14.21) *Let (X, d) be a sequentially compact metric space, and let $\epsilon > 0$. Then there exists a finite $\epsilon-$net for X*

Theorem 201 (14.22) *Any sequentially compact metric space X is compact*

Remark: Any sequentially compact metric space is bounded

Remark: A closed subset of a sequentially compact metric space is sequentially compact

Remark: A sequentially compact subspace of a metric space X is closed in X

Remark: The product of two sequentially compact metric spaces is sequentially compact.

15 Uniform convergence

Definition 202 (16.1) *The sequence (f_n) converges to f pointwise on D if for each $x \in D$ the real number sequence $(f_n(x))$ converges to $f(x)$*

Definition 203 (16.3) *A sequence (f_n) of real valued functions defined on a domain $D \subseteq \mathbb{R}$ converges to a function f uniformly on D if given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and all $x \in D$*

Proposition 204 (16.4) *Let $f, f_n : D \to \mathbb{R}$ be real valued functions on D. Then $f_n \to f$ uniformly on D if $M_n = \sup_{x \in D} |f_n(x) - f(x)|$ exists for all sufficiently large n and $M_n \to 0$ as $n \to \infty$*

15.1 Cauchy's criterion

Definition 205 (16.8) *A sequence (f_n) of real valued functions defined on a domain $D \subseteq \mathbb{R}$ is said to be uniformly Cauchy on D if given $\epsilon > 0$ there exists an integer N such that $|f_m(x) - f_n(x)| < \epsilon$ for all $m, n \geq N$ and all $x \in D$*

Theorem 206 (16.9 Cauchy's criterion for uniform convergence) *Let (f_n) be a sequence of real valued functions defined on $D \subseteq \mathbb{R}$. Then (f_n) converges uniformly on D if and only if it is uniformly Cauchy on D*

Remark: The Cauchy's criterion for uniform convergence has the advange that the limit function need not to be known in advance in order to prove uniform convergence

15.2 Uniform limits of sequences

Theorem 207 (16.10) *if $f_n : (a, b) \to \mathbb{R}$ is continuous at $c \in (a, b)$ for every $n \in \mathbb{N}$ and if $f_n \to f$ uniformly on (a, b) then f is continuous at c*

Corollary 208 (16.11) *Suppose for each $n \in \mathbb{N}$ the function $f_n; [a, b] \to \mathbb{R}$ is continuous, and that (f_n) converges to a function f uniformly on $[a, b]$. Then f is continuous on $[a, b]$*

Corollary 209 (16.12) *Suppose that the pointwise limit of a sequence (f_n) of continuous functions on $[a, b]$ is not continuous on $[a, b]$. Then the convergence is not uniform*

16 Complete metric space

Definition 210 (17.2) *A metric space X is complete if every Cauchy sequence in X converges (to a point of X)*

Remark: $\mathbb{R}$ is complete. $\mathbb{Q}$ is not complete. $(0,1) \subset \mathbb{R}$ is not complete

Proposition 211 (17.4) *suppose that X, Y are metric spaces and there exists a bijective map $f : X \to Y$ such that both f and f^{-1} are uniformly continuous. Then X is complete if and only if Y is complete*

Proposition 212 (17.6) *A complete subspace Y of a metric space X is closed in X*

Proposition 213 (17.7) *A closed subspace Y of a complete metric space X is complete*

Proposition 214 (17.9) *Any compact metric space X is complete*

Lemma 215 (17.10) *If a Cauchy sequence (x_n) in a metric space X has a subsequence converging to $x \in X$ then (x_n) converges to x*

Proposition 216 (17.11) *The product of two metric spaces (X, d_X) and (Y, d_Y) is complete if and only if (X, d_X) and (Y, d_Y) are complete*

Corollary 217 (17.12) *The product of a finite number of metric spaces is complete if and only if all the factors are complete*

Corollary 218 (17.13) $\mathbb{R}^n$ *is complete for each $n \in N$*

16.1 Banach's fixed point theorem

Definition 219 (17.18) *Given any sel-map $f : S \to S$ of a set S, a fixed point of f is a point $p \in S$ such that $f(p) = p$*

Definition 220 (17.19) *For given positive real numbers α and K, a function $f : D \to \mathbb{R}$ satisfies a Lipschitz condition of order α on D, with constant K if*

$$|f(x) - f(y)| \leq K|x - y|^{\alpha} \quad \forall x, y \in D$$

Proposition 221 (17.20)

- *a if f satisfies a Lipschitz condition of order $\alpha > 0$ on D then f is uniformly continuous on D*
- *b if f satisfies a Lipschitz condition of order $\alpha > 1$ on $[a, b]$ then f is constant on $[a, b]$*
- *c if $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) with $|f'(x)| \leq K$ for all $x \in (a, b)$ then f satisfies a Lipschitz condition of order 1 with constant K on $[a, b]$*

Theorem 222 (17.22 special case of Banach's fixed point theorem) *if $f : [a, b] \to [a, b]$ satisfies a Lipschitz condition of order 1 with constant $K < 1$ on $[a, b]$ then f has a unique fixed point p in $[a, b]$. Moreover, if x_1 is any point in $[a, b]$ and $x_n = f(x_{n-1})$ for $n > 1$, then (x_n) converges to p. The same result holds if $[a, b]$ is replaced throughout by $(-\infty, b]$ or $[a, \infty)$*

16.1.1 Contraction mappings

Definition 223 (17.24) *Let (X, d) be a metric space. A map $f : X \to X$ is a contraction if for some constant $K < 1$ we have $d(f(x), f(y)) \leq Kd(x, y)$ for all $x, y \in X$*

Lemma 224 (17.25) *Any contraction of a metric space X is uniformly continuous*

Theorem 225 (17.26 Banach's fixed point theorem) *if $f : X \to X$ is a contraction of a complete metric space X then f has a unique fixed point p in X*

16.1.2 Applications of Banach's fixed point theorem

Theorem 226 (17.29) *Suppose $K : [a, b] \times [a, b] \to \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ are continuous. Then the Volterra equation*

$$\phi(x) = f(x) + \int_a^x K(x, y)\phi(y)\, dy$$

has a unique continuous solution ϕ on $[a, b]$

Theorem 227 (17.31) *Supppose that $f : D = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b] \to \mathbb{R}$ is continuous and satisfies the Lipschitz condition*

$$|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2| \quad \forall (x, y_1), (x, y_2) \in D, \text{ for some } K > 0$$

Let M be an upper bound for $|f(x, y)|$ on D, and let $c = min\{a, b/M\}$. Then on $I = [x_0 - c, x_0 + c]$ there exists a unique solution y of the differetial equation $\frac{dy}{dx} = f(x, y)$ such that $y(x_0) = y_0$